Binary Morphisms to Ultimately Periodic Words

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Abstract

This paper classifies morphisms from $\{0, 1\}$ that map to ultimately periodic words. In particular, if a morphism h maps an infinite non-ultimately periodic word to an ultimately periodic word then it must be true that $h(0)$ commutes with $h(1)$.

1 Introduction

In this short note we present a rough solution to an open problem in the study of combinatorics on words, due to Jean-Paul Allouche [4]. We omit a full discussion of the subject matter in this manuscript; there are numerous excellent texts that provide background on the subject of combinatorics on words, and specifically the study of morphisms [1, 2]. The problem of interest is the following:

Problem 1.1. *Let w be an infinite word over $\{0, 1\}$ that is not ultimately periodic, and let h be a morphism. Suppose $h(w)$ is ultimately periodic. Prove (or disprove) that $h(0)$ commutes with $h(1)$.*

This problem has applications to continued fraction expansions [3]. In this paper we shall prove that, indeed, $h(0)$ must commute with $h(1)$.

A few short notes on notation: given a string x, we shall use x^ω to denote the infinite repetition of x (i.e. the word $xxx\cdots$). Also, we shall write $x[i]$ to mean the character of x at index i, and $x[i, j]$ to mean the substring of x consisting of those characters at indexes i to j, inclusive. For example, if x is the binary word 0100110 then $x[2, 5] = 1001$.

2 Main Result

Before proving the main result, we require a simple proposition regarding injective morphisms.

Proposition 2.1. *Suppose h is a morphism from $\{0, 1\}$ such that $h(0)$ and $h(1)$ do not commute. Then for all $a, b \in \{0, 1\}^*$, $a \neq b \implies h(a) \neq h(b)$.*

Proof. Assume that h maps to an alphabet that does not contain 0 or 1. Denote $x = h(0)$ and $y = h(1)$. First note that $h(01) \neq h(10)$ implies that $x \neq y$. Also, if we had $k := |x| = |y|$, then given $h(a)$ we could uniquely determine a simply by matching the letters of $h(a)$ to the letters of x and y, k at a time. Being able to uniquely determine a from $h(a)$ would imply that $a \neq b \implies h(a) \neq h(b)$, as required. So we can assume $|x| \neq |y|$.

Let $n = \max\{|x|, |y|\}$. Assume without loss of generality that $|y| > |x|$, so we have $n = |y|$. We now proceed by induction on n.

If $n = 1$ then we must have $x = \epsilon$. But then we would have $xy = y = yx$, a contradiction.

If $n = 2$ then $|x| = 1$ and $|y| = 2$. But since x and y do not commute, we cannot have $y = xx$. Let c be the first letter that appears in y that is not x. Now, given a string $h(a)$, we can uniquely determine the value of a as follows. Scan the word $h(a)$ from left to right. Every time we find a c, we map it plus the preceeding character (if $y = xc$) or the following character (otherwise) to 1. Once that is done, map all of the remaining x characters to 0. This process generates the string a in a unique way. We conclude that if $a \neq b$ then we must have $h(a) \neq h(b)$.

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This concludes the base cases.

So suppose now that $\max\{|x|, |y|\} = n > 2$. Suppose also for contradiction that there exist binary words a, b such that $a \neq b$ but $h(a) = h(b)$. Since $h(a) = h(b)$ and neither x nor y is ϵ it cannot be the case that either a or b is a prefix of the other. There must therefore be some minimal index $i \leq \min\{|a|, |b|\}$ such that $a[i] \neq b[i]$. But then if we let $z = h(a[1, i-1])$ we have that both zx and zy are prefixes of $h(a)$. We conclude that x is a prefix of y. Say $y = xy'$, with $|y'| < |y|$.

Let $f\colon \{0,1\}^* \to \{0,1\}^*$ be the morphism $f(0) = 0$, $f(1) = 01$. Let h be the morphism on $\{0,1\}^*$ given by $h(0) = x$, $h(1) = y'$. Note that $h = g \circ f$. By our base case, $a \neq b \implies f(a) \neq f(b)$. Also, $\max\{|x|, |y'|\} < |y| = n$, so by induction we must now have

$$h(a) = g(f(a)) \neq g(f(b)) = h(b) \tag{1}$$

as required. □

We are now ready to prove the main result.

Theorem 2.2. *Suppose w is an infinite word over $\{0,1\}$ that is not ultimately periodic, and let h be a morphism. If $h(w)$ is ultimately periodic then $h(0)$ commutes with $h(1)$.*

Proof. Suppose for contradiction that $h(0)$ does not commute with $h(1)$. Since $h(w)$ is ultimately periodic, we can write

$$h(w) = yz^{\omega} \tag{2}$$

for finite strings y and z.

Note that every prefix of w must map to a prefix of yz^{ω}, so in particular there must be infinitely many prefixes of w that map to a string of the form $yz^*z[1,k]$ for any $k \leq |z|$. But there are only finitely many possible values for k. There must therefore be some prefix z_1 of z such that infinitely many prefixes of w map to strings of the form yz^*z_1 for any $t \geq 0$.

Now say $z = z_1z_2$. Say that $x, xa_1, xa_1a_2, \ldots$ are the infinitely many prefixes of w discussed above, where each a_i is chosen to minimize $|a_i|$. Then we have

$$h(a_i) = z_2z^{p_i}z_1 \ \forall i \geq 1 \tag{3}$$

where $p_i \geq 0$ for all $i \geq 1$.

Suppose first that all p_i are equal. Then since $h(0)$ doesn't commute with $h(1)$, Proposition 2.1 tells us that since all $h(a_i)$ are equal, all a_i must be equal. But then $w = xa_1a_1a_1\cdots$ is ultimately periodic, a contradiction.

Suppose instead not all p_i are equal, so there is some i such that $p_i \neq p_{i+1}$. Then we have

$$\begin{aligned} &h(a_ia_{i+1}) \\ =&z_2z^{p_i}z_1z_2z^{p_{i+1}}z_1 \\ =&z_2z^{p_{i+1}}z_1z_2z^{p_i}z_1 \\ =&h(a_{i+1}a_i) \end{aligned} \tag{4}$$

So by Proposition 2.1, $a_ia_{i+1} = a_{i+1}a_i$.

The second theorem of Lyndon and Schutzenberger now tells us that there exists some b such that $a_i = b^k$, $a_{i+1} = b^l$. Since we assumed $p_i \neq p_{i+1}$ we must have $k \neq l$. If $k < l$ then $|a_i| < |a_{i+1}|$ and a_{i+1} has a_i as a strict prefix. But a_i is of the form $z_2z^*z_1$, so this contradicts the assumed minimality of $|a_{i+1}|$. If $k > l$ then an identical argument contradicts the minimality of $|a_i|$. □

References

[1] J.-P. Allouche, J. O. Shallit, *Automatic Sequences: Theory, Applications, Generalizations*, Cambridge University Press, 2003.

[2] M. Lothaire, *Combinatorics on Words*, vol. 17 of Encyclopedia of Mathematics and its Applications, Addison-Wesley, Reading, Massachusetts, 1983.

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[4] J. O. Shallit, personal communication, August 2004.